MOTION OF A SPHERICAL PISTON WITH CONSTANT VELOCITY IN AN INHOMOGENEOUS MEDIUM

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We shall investigate the motion of the gas behind a spherical piston which moves with constant velocity in a medium where the density varies according to the law

$$\rho = \rho_1 [1 - \varepsilon z^{\varkappa}] \tag{1}$$

where z is a Cartesian coordinate, ϵ is a small parameter, ρ_1 and κ are constants.

The analogous problem for a strong explosion was investigated by Karlikov [1]. We shall take spherical coordinates r, θ and ϕ in space. According to the conditions in the problem, the pressure p, density ρ, velocity components v_r and v_θ and entropy S do not depend on the coordinate ϕ and the velocity coordinate $v_\phi = 0$. All these physical quantities are functions of the variables t, r, θ and of parameters ρ_1, p_1, ϵ, κ, $\gamma = c_p/c_v$. From these quantities we may form only three dimensionless variables

$$\lambda = \frac{\gamma p_1}{\rho_1} \frac{t^2}{r^2}, \qquad \mu = \varepsilon r^{\varkappa}, \qquad \theta$$

In this manner the desired dimensional functions may be represented in terms of dimensionless functions which depend upon the dimensionless variables

$$v_r = \frac{r}{t} V_r'(\lambda, \mu, \theta), \qquad v_\theta = \frac{r}{t} V_0'(\lambda, \mu, \theta),$$
$$p = \rho_1 \left(\frac{r}{t}\right)^2 P'(\mu, \theta), \qquad \rho = \rho_1 R'(\lambda, \mu, \theta) \tag{2}$$

The problem of the gas motion behind a piston moving with constant velocity in a homogeneous medium was solved by Sedov [2]. Let

$V_0(\lambda)$, $P_0(\lambda)$, $R_0(\lambda)$ be the solutions of this problem. Then we shall represent the desired linearized solutions in the form

$$V_r' = V_0(\lambda) + V_r^\circ(\lambda, \theta), \qquad V_\theta' = \mu V_\theta^0(\lambda, \theta), \qquad P' = P_0(\lambda) + \mu P^\circ(\lambda, \theta),$$
$$R' = R_0(\lambda) + \mu R^\circ(\lambda, \theta) \tag{3}$$

The basic equations in spherical coordinates have the form

$$r^2 \frac{\partial \rho}{\partial t} + \frac{\partial r^2 \rho v_r}{\partial r} + \frac{r}{\sin\theta} - \frac{\partial \rho v_\theta \sin\theta}{\partial \theta} = 0 \qquad \frac{\partial S}{\partial t} + v_r \frac{\partial S}{\partial r} + \frac{v_\theta}{r}\frac{\partial S}{\partial \theta} = 0$$
$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{r_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{4}$$
$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0$$

After transformation to the dimensionless variables and variation with respect to μ we obtain the system of partial differential equations for V_r°, V_θ°, P°, R°. This system of partial differential equations may be reduced to a system of ordinary differential equations if use is made of the Fourier method. We shall express the desired functions in the form

$$V_r^\circ = F(\theta) V_r(\lambda), \qquad P^\circ = F(\theta) P(\lambda), \qquad R^\circ = F(\theta) R(\lambda), \qquad V_\theta^\circ = N(\theta) V_\theta(\lambda) \tag{5}$$
$$\left(F(\theta) = P_\nu(\cos\theta), \qquad N(\theta) = -\frac{1}{n^2}\frac{dP_\nu}{d\theta} \right)$$

where $P(\cos\theta)$ is a Legendre polynomial and ν is an integer. If we assume that $V_\theta = n^2 V^{(\nu)}(\lambda)$, then the solutions for V_r°, P°, R° and V_θ° may be looked for in the form of infinite series of the following form:

$$V_r^\circ = \sum_{\nu=0}^{\infty} P_\nu(\cos\theta) V_r^{(\nu)}(\lambda) \qquad R^\circ = \sum_{\nu=0}^{\infty} P_\nu(\cos\theta) R^{(\nu)}(\lambda),$$
$$P^\circ = \sum_{\nu=0}^{\infty} P_\nu(\cos\theta) P^{(\nu)}(\lambda), \qquad V_\theta^\circ = \sum_{\nu=0}^{\infty} (-1) \frac{dP_\nu}{d\theta} V_\theta^{(\nu)}(\lambda) \tag{6}$$

We shall investigate the boundary conditions. At the piston the normal gas velocity is equal to the velocity of the piston $v_r = u$, and since $V_0 = 1$, then $V_r(\lambda_n) = 0$; λ_n is the value of λ on the piston and u is the velocity of the piston.

At some distance in front of the piston there is a shock-wave. To simplify the problem we shall assume that the velocity of the piston is large and the conditions at the shock-wave will be accounted for in the form

$$v_2 = \frac{2}{\gamma+1} c, \qquad \rho_2 = \frac{\gamma+1}{\gamma-1}\rho, \qquad p_2 = \frac{2}{\gamma+1}\rho c^2 \tag{7}$$

where c is the velocity of the shock-wave.

The conditions at the shock-wave after transformation to dimensionless variables and after variation with respect to μ have the form

$$V_{r2}{}^{\circ}(\lambda^*, \theta) = 2\left[\frac{\varkappa+1}{\gamma+1} - V_0(\lambda^*) + 2\lambda^*\left(\frac{dV_0}{d\lambda}\right)_{\lambda=\lambda^*}\right] f(\theta)$$
$$V_{\theta 2}{}^{\circ}(\lambda^*, \theta) = -\frac{2}{\gamma+1} f'(\theta) \tag{8}$$
$$R_2{}^{\circ}(\lambda^*, \theta) = 2\lambda^*\left(\frac{dR_0}{d\lambda}\right)_{\lambda=\lambda^*} f(\theta) - \frac{\gamma+1}{\gamma-1}\cos^{\varkappa}\theta$$
$$P_2{}^{\circ}(\lambda^*, \theta) = \left[4\frac{\varkappa+1}{\gamma+1} - 2P_0(\lambda^*) + 2\lambda^*\left(\frac{dP_0}{d\lambda}\right)_{\lambda=\lambda^*}\right] f(\theta) - \frac{2}{\gamma+1}\cos^{\varkappa}\theta$$

where λ^* is the value of λ at the shock-wave. The radius vector r_2 of the shock-wave will be represented in the form

$$r_2 = r_{20}[1 + \mu^* f(\theta)] \tag{9}$$

where $\mu^* = \epsilon r_{20}{}^{\kappa}$ and $f(\theta)$ is an unknown function. As is well known, for automodel motion $V_0(\lambda)$, $P_0(\lambda)$, $R_0(\lambda)$ cannot be expressed in analytical form. After transformation to the independent variable V_0 we find an approximate solution by expanding the desired functions in powers of $(1 - V_0)$

$$\lambda = \lambda_{00}\left[1 - \tfrac{2}{3}(1 - V_0) + \ldots\right], \qquad P_0 = P_{00}\left[1 - \tfrac{2}{3}(1 - V_0) + \ldots\right]$$
$$R_0 = R_{00}\left[1 + \tfrac{1}{3}(1 - V_0)^2 + \ldots\right] \tag{10}$$

where λ_{00}, P_{00} and R_{00} are the values of the functions on the piston. The system of ordinary differential equations for the functions $V_r{}^{(\nu)}(V_0)$, $P^{(\nu)}(V_0)$, $R^{(\nu)}(V_0)$, $V_\theta{}^{(\nu)}(V_\theta)$ has a singular point at $V_0 = 1$. We shall look for the solutions in the form of series of powers of $(1 - V_0)$:

$$V_r{}^{(\nu)} = (1 - V_0)^s \sum_{n=0}^{\infty} a_{ni}{}^{(\nu)}(1 - V_0)^n, \qquad R^{(\nu)} = (1 - V_0)^s \sum_{n=0}^{\infty} c_{ni}^{(\nu)}(1 - V_0)^n$$
$$P^{(\nu)} = (1 - V_0)^s \sum_{n=0}^{\infty} b_{ni}{}^{(\nu)}(1 - V_0)^n, \qquad V_\theta{}^{(\nu)} = (1 - V_0)^s \sum_{n=0}^{\infty} d_{ni}{}^{(\nu)}(1 - V_0)^n \tag{11}$$

The characteristic equation of the system will be the following:

$$s^2\left[\frac{1}{2}\frac{\varkappa}{\lambda_n}\left(\frac{d\lambda}{dV_0}\right)_{V_0=1} - s\right]\left[\frac{\varkappa+1}{2\lambda_n}\left(\frac{d\lambda}{dV_0}\right)_{V_0=1} - s\right] = 0 \tag{12}$$

The roots of this equation, if (10) is taken into account, have the following values:

$$s_1 = 0, \qquad s_2 = 0, \qquad s_3 = \tfrac{1}{3}\varkappa, \qquad s_4 = \tfrac{1}{3}(\varkappa + 1)$$

The roots s_1 and s_2 denote the solutions with a logarithmic singularity [3]. If s_3 and s_4 are not integers, the solutions for $V_r^{(\nu)}$, $P^{(\nu)}$, $R^{(\nu)}$, $V_\theta^{(\nu)}$, neglecting $(1-V_0)^2$ may be represented in the following form:

$$V_r^{(\nu)} = \frac{b_{01}^{(\nu)}}{3\gamma P_{00}}\left[\varkappa + \frac{\nu(\nu+1)\gamma P_{00}}{(\varkappa+1)R_{00}}\right](1-V_0)[-\ln(1-V_0)] - \frac{\nu(\nu+1)}{\varkappa+4}d_{04}^{(\nu)}(1-V_0)^{(\varkappa+4)/3} \tag{13}$$

$$P^{(\nu)} = b_{01}^{(\nu)}\left\{\left[1-\frac{\varkappa+2}{3}(1-V_0)\right]\left[\ln(1-V_0)+1\right]+\frac{6+\varkappa}{3}(1-V_0)\right\}$$

$$R^{(\nu)} = \frac{R_{00}}{\gamma P_{00}}b_{01}^{(\nu)}\left\{\left[1-\frac{\varkappa}{3}(1-V_0)\right]\left[\ln(1-V_0)+1\right]+\frac{3\varkappa+12}{(3-\varkappa)}(1-V_0)\right\} + c_{03}^{(\nu)}(1-V_0)^{\varkappa/3}$$

$$V_\theta^{(\nu)} = \frac{b_{01}^{(\nu)}}{R_{00}(\varkappa+1)}\left\{\left[1-\frac{\varkappa+2}{3}(1-V_0)\right]\left[\ln(1-V_0)+1\right] + \frac{(3\varkappa+12)}{3(2-\varkappa)}(1-V_0)\right\} + d_{04}^{(\nu)}(1-V_0)^{(\varkappa+1)/3}\left[1-\frac{1}{3}(1-V_0)\right]$$

where the condition at the piston is already accounted for. The unknown constants $d_{04}^{(\nu)}$, $c_{03}^{(\nu)}$, $b_{01}^{(\nu)}$ are determined from the conditions at the shock-wave. In the case $\kappa = 1$ the solutions are as follows:

$$v_r = \frac{r}{t}\left[V_0 + \mu\left\{\frac{b_{01}^{(1)}}{3\gamma P_{00}}\left(1+\frac{\gamma P_{00}}{R_{00}}\right)(1-V_0)[-\ln(1-V_0)] - \frac{2}{5}d_{04}^{(1)}(1-V_0)^{5/3}\right\}\cos\theta\right]$$

$$v_\theta = \frac{r}{t}\mu\left[\frac{b_{01}^{(1)}}{2R_{00}}\{(V_0)\ln(1-V_0)+1+4(1-V_0)\}+d_{04}^{(1)}(1-V_0)^{2/3}\left[1-\frac{1}{3}(1-V_0)\right]\right]\sin\theta$$

$$p = \rho_1\left(\frac{r}{t}\right)^2\left\{P_{00}\left[1-\frac{2}{3}(1-V_0)\right] + \mu b_{01}^{(1)}\left[V_0\ln(1-V_0)+1+\frac{4}{3}(1-V_0)\right]\cos\theta\right\} \tag{14}$$

$$\rho = \rho_1\left[R_{00} + \mu\left(\frac{b_{01}^{(1)}R_{00}}{\gamma P_{00}}\left\{\left[1-\frac{1}{3}(1-V_0)\right]\ln(1-V_0)+1+\frac{13}{6}(1-V_0)\right\} + c_{03}^{(1)}(1-V_0)^{1/3}\right)\cos\theta\right]$$

$$r_2 = r_{20}[1+\mu^* c_1\cos\theta]$$

In the case $\kappa = 2$ the root s_4 will be an integer. In that case

$$V_r^{(\nu)} = \frac{b_{01}^{(\nu)}}{3\gamma P_{00}}\left[-2+\frac{\nu(\nu+1)}{3}\frac{\gamma P_{00}}{R_{00}}\right](1-V_0)[-\ln(1-V_0)]$$

$$P^{(\nu)} = b_{01}^{(\nu)}\left\{\left[1-\frac{4}{3}(1-V_0)\right]\ln(1-V_0)+1+\frac{4}{3}(1-V_0)\right\}$$

$$R^{(\nu)} = \frac{b_{01}^{(\nu)}R_{00}}{\gamma P_{00}}\left\{\left[1-\frac{2}{3}(1-V_0)\right]\ln(1-V_0) + \frac{16}{3}(1-V_0)\right\} + c_{03}^{(\nu)}(1-V_0)^{2/3} \tag{15}$$

$$V_\theta^{(\nu)} = \left[\frac{b_{01}^{(\nu)}}{3R_{00}} + d_{11}^{(\nu)}(1-V_0)\right][1+\ln(1-V_0)]$$

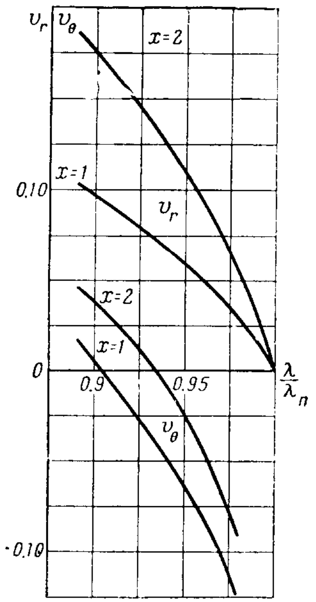

Fig. 1.

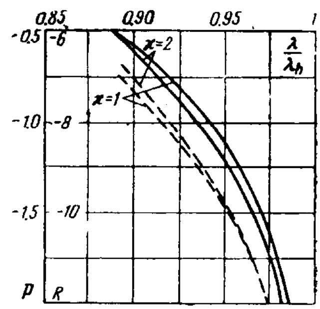

Fig. 2.

The final formulas for the solution will have the following form:

$$v_r = \frac{r}{t}\left\{V_0 + \mu\left[\frac{b_{01}^{(0)}}{3\gamma P_{00}}\left(2\,\frac{\gamma P_{0n}}{3R_{0n}}\right) + \frac{b_{01}^{(2)}}{3\gamma P_{0n}}\left(1+\frac{\gamma P_{00}}{R_{00}}\right)(3\cos^2\theta - 1)\right]\times\right.$$
$$\left.\times(1-V_0)\,[-\ln(1-V_0)]\right\} \qquad (16)$$

$$v_\theta = \frac{r}{t}\,\mu\left[\frac{b_{01}^{(2)}}{3R_{00}} + d_{11}^{(2)}(1-V_0)\right]3\cos\theta\sin\theta\,[\ln(1-V_0)+1]$$

$$p = \rho_1\left(\frac{r}{t}\right)^2\left(P_{00}\left[1-\tfrac{2}{3}(1-V_0)\right] + \mu\left[b_{01}^{(0)} + \tfrac{1}{2}b_{01}^{(2)}(3\cos^2\theta - 1)\right]\times\right.$$
$$\left.\times\left\{\left[1-\tfrac{4}{3}(1-V_0)\right]\ln(1-V_0) + 1 + \tfrac{4}{3}(1-V_0)\right\}\right)$$

$$\rho = \rho_1\left(R_{0n} + \mu\,\frac{R_{00}}{\gamma P_{00}}\left[b_{01}^{(0)} + \tfrac{1}{2}b_{01}^{(2)}(3\cos^2\theta - 1)\right]\times\right.$$
$$\times\left\{\left[1-\tfrac{2}{3}(1-V_0)\right]\ln(1-V_0) + 1 + \tfrac{16}{3}(1-V_0)\right\} +$$
$$\left.+\,\mu\left\{\left[c_{03}^{(0)} + \tfrac{1}{2}c_{03}^{(2)}(3\cos^2\theta - 1)\right](1-V_0)^{2/3}\right)\right.$$

$$r_2 = r_{20}\left\{1 + \mu^*\left[c_0 + \tfrac{1}{2}c_2(3\cos^2\theta - 1)\right]\right\}$$

Results of the calculation of V_r° as a function of λ/λ_n for $\theta = 0$ and V_θ° for $\theta = 45^\circ$ are presented in the form of graphs in Fig. 1; Fig. 2

shows the variation of *R* by solid lines and that of *p* by dashed lines.

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